Beam Columning in Ground Wind Load Analyses

Robert P. McFarland*
Martin Company, Baltimore, Md.

The determination of the effects of beam columning in the ground wind load analyses of launch vehicles is presented. Considering a uniform vehicle, it first is demonstrated that the exact solution can be simplified adequately by a power series. Subsequently, an approximation is shown which is accurate enough for any practical application. The increase in base bending moment induced by the beam columning effects is presented graphically as a function of nondimensional vehicle parameters. It is shown that present-day large-liquid boosters experience an increase in base bending moments of approximately 18%. The significance of base flexibility is discussed briefly, and an approximate relationship is established between beam columning on the launch pad and in powered flight.

The determination of the bending moments on a launch vehicle exposed to a ground wind generally consists of evaluating the steady-state drag moments in the direction of the wind and superimposing some incremental moment associated with wind-induced oscillations. This has been accomplished using either a discrete ground wind field[1] or wind spectra.[2] Little attention has been paid to the increase in bending moment associated with the lateral displacements of the vehicle's distributed mass, caused by the interaction of the applied loading and the flexibility of the structure. In some instances, this can be greater than the dynamic effects.

The present note evaluates this "beam column" action by considering a cantilevered vertical uniform beam attached to a flexible base, with some initial misalignment. Experience has shown that the determination of second-order effects, such as beam columning, by a uniform beam analogy yields results that are within 10% of those calculated with a nonuniform launch vehicle.

In the presence of a load-inducing media, such as a ground wind, launch vehicles experience lateral deformations that increase the bending moments associated with the applied loading, because of the eccentricity of the displaced beam in relation to the vertical, as shown in Fig. 1a. These deformations and, consequently, the bending moments are increased further by any misalignment of the beam from the vertical, denoted by the angle ϕ_0, and by the rigidity of the base, denoted by the torsional spring constant K_ϕ.

From Fig. 1b, the differential equation of the deflection curve is

$$EIy''(x) = \frac{Fx^2}{2L} + \frac{W}{L}\left[\int_0^x y(\lambda)d\lambda - xy(x)\right] \tag{1}$$

in which EI is the constant bending stiffness, F the applied force, W the weight, and L the length. Differentiating Eq. (1) with respect to x and introducing the notation $U(x/L) = y'(x)$, one has

$$U'' + \frac{WL^2}{EI}\left(\frac{x}{L}\right)U = \frac{FL^2}{EI}\left(\frac{x}{L}\right) \tag{2}$$

The solution of this equation can be effected by Bessel functions as

$$U = \frac{F}{W} + A\left(\frac{x}{L}\right)^{1/2} J_{1/3}\left\{\frac{2}{3}\left[\frac{WL^2}{EI}\left(\frac{x}{L}\right)^3\right]^{1/2}\right\} + B\left(\frac{x}{L}\right)^{1/2} J_{-1/3}\left\{\frac{2}{3}\left[\frac{WL^2}{EI}\left(\frac{x}{L}\right)^3\right]^{1/2}\right\} \tag{3}$$

in which A and B are arbitrary constants and $J_{1/3}$ and $J_{-1/3}$ are Bessel functions of the first kind of order $\frac{1}{3}$ and $-\frac{1}{3}$, with real argument $\frac{2}{3}[(WL^2/EI)(x/L)^3]^{1/2}$.

The arbitrary constants may be determined by application of the boundary conditions that $U' = 0$ at $x/L = 0$ and $U = -[M(L)/K\phi] - \phi_0$ at $x/L = 1$. Upon evaluating these constants, Eq. (3) becomes

$$\frac{M(x/L)}{(F + W\phi_0)(L/2)} = \frac{\left(\frac{x}{L}\right)\left[\frac{2}{(WL^2/EI)^{1/2}}\cdot\frac{J_{2/3}\{\frac{2}{3}[(WL^2/EI)(x/L)^3]^{1/2}\}}{J_{-1/3}[\frac{2}{3}(WL^2/EI)^{1/2}]}\right]}{1 - (WL/2K_\phi)\left[\frac{2}{(WL^2/EI)^{1/2}}\cdot\frac{J_{2/3}[\frac{2}{3}(WL^2/EI)^{1/2}]}{J_{-1/3}[\frac{2}{3}(WL^2/EI)^{1/2}]}\right]} \tag{4}$$

where $(F + W\phi_0)(L/2)$ is the rigid body bending moment at the base. Since only the aft end of most launch vehicles is designed by the ground wind condition, only the base of the vehicle, i.e., $x/L = 1$, will be of concern here.

Before proceeding with the development of this equation, consider the bracketed term of Eq. (4). Certain values of the argument $\frac{2}{3}(WL^2/EI)^{1/2}$ will yield the zero of $J_{-1/3}[\frac{2}{3}(WL^2/EI)^{1/2}]$. These critical values correspond to the lateral buckling of the vertical beam under its own weight. The smallest value of this argument at which buckling will occur, obtained with the aid of Ref. 3, is $\frac{2}{3}(WL^2/EI)^{1/2} = 1.86635$. Defining W_{CR} as the critical buckling weight, one has

$$W_{CR} = 7.83734EI/L^2 \tag{5}$$

This result is in agreement with that obtained by Timoshenko.[4]

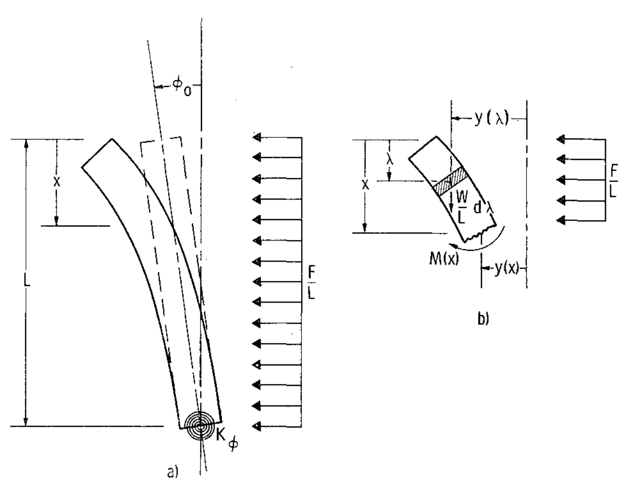

Fig. 1 Vertical uniform beam in a constant wind field, flexible base with initial misalignment.

Received April 3, 1963. The author wishes to express his gratitude to J. M. Hedgepeth of the Martin Company for his valuable suggestions.

* Staff Engineer, Structures and Materials Department. Member AIAA.

Equation (4), in its present form, is inconvenient to handle. Since vehicles whose weight approaches that defined by Eq. (5) are not of concern here, it is permissible to expand the Bessel functions in the bracketed terms of Eq. (4) into a limited number of terms by a power series. Letting $x/L = 1$, one has

$$\frac{2}{(WL^2/EI)^{1/2}} \cdot \frac{J_{2/3}[\frac{2}{3}(WL^2/EI)^{1/2}]}{J_{-1/3}[\frac{2}{3}(WL^2/EI)^{1/2}]} = 1 + \frac{1}{10}\left(\frac{WL^2}{EI}\right) + \frac{1}{80}\left(\frac{WL^2}{EI}\right)^2 + \frac{7}{4400}\left(\frac{WL^2}{EI}\right)^3 + \frac{1}{4928}\left(\frac{WL^2}{EI}\right)^4 + \ldots \quad (6)$$

To check the adequacy of the number of terms, let $WL^2/EI = 2$. This corresponds to a vehicle whose weight is almost $\frac{1}{4}$ of the buckling weight defined by Eq. (5), a practical maximum. The left side of Eq. (6), to five significant figures, yields 1.2672. Upon substitution into the right-hand side, one obtains 1.2660. Excluding the rigid body term, the error introduced is 0.2660/0.2672, or less than 0.5% on the low side, which is accurate enough for any practical application.

Now that the power series has been justified, one may go one step further in the simplification of Eq. (4), and that is the substitution of Eq. (5) into Eq. (6), or

$$\frac{2}{(WL^2/EI)^{1/2}} \cdot \frac{J_{2/3}[\frac{2}{3}(WL^2/EI)^{1/2}]}{J_{-1/3}[\frac{2}{3}(WL/EI)^{1/2}]} = 1 + 0.783734\frac{W}{W_{CR}}\left[1 + 0.979668\frac{W}{W_{CR}} + 0.977199\left(\frac{W}{W_{CR}}\right)^2 + 0.976867\left(\frac{W}{W_{CR}}\right)^3 + \ldots\right] \quad (7)$$

Noting that the series of terms on the right side may be approximated by $1/[1 - (W/W_{CR})]$, one has

$$\frac{2}{(WL^2/EI)^{1/2}} \cdot \frac{J_{2/3}[\frac{2}{3}(WL^2/EI)^{1/2}]}{J_{-1/3}[\frac{2}{3}(WL^2/EI)^{1/2}]} \cong 1 + \frac{0.7837}{(W_{CR}/W) - 1} \quad (8)$$

Checking the adequacy of this approximation as before, i.e., with $WL^2/EI = 2$, one finds the value of the magnification factor $0.7837/[(W_{CR}/W) - 1]$ to be 0.2685. Compared to the exact value of 0.2672 (noted previously), this gives an error of less than 0.5% on the high side. Thus, the two approximations have bounded the exact value by the same amount. The adequacy of Eq. (8) for other values of WL^2/EI is shown in Table 1.

Substituting Eq. (8) into Eq. (4), the base bending moment now becomes

$$\frac{M(L)}{(F + W\phi_0)(L/2)} \cong \frac{1 + [0.784/(W_{CR}/W) - 1]}{1 - (WL/2K_\phi)\ \{1 + [0.784/(W_{CR}/W) - 1]\}} \quad (9)$$

To illustrate the influence of beam columning and base flexibility on the base bending moment, as defined by Eq. (9), Fig. 2 presents the increase in moment over the rigid base

Table 1 Comparison of exact solution with approximate solution

$\frac{WL^2}{EI}$	$\frac{2}{(WL^2/EI)^{1/2}} \cdot \frac{J_{2/3}[\frac{2}{3}(WL^2/EI)^{1/2}]}{J_{-1/3}[\frac{2}{3}(WL^2/EI)^{1/2}]}$	$\frac{0.7837}{(W_{CR}/W - 1)}$
2.00	1.2672	1.2685
1.50	1.1845	1.1858
1.00	1.1143	1.1146
0.75	1.0828	1.0830
0.40	1.0421	1.0422
0.25	1.0258	1.0259
0.15	1.0153	1.0153
0.08	1.0081	1.0081

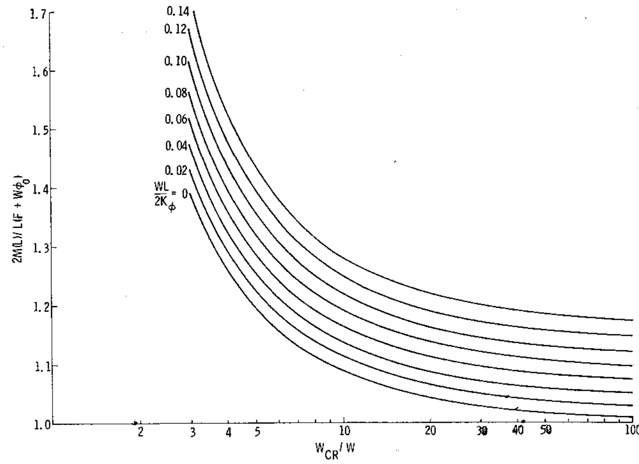

Fig. 2 Increase in base bending moment as a function of W_{CR}/W and $WL/2K_\phi$.

moment $(F + W\phi_0)(L/2)$ as a function of the ratio W_{CR}/W for a range of the base flexibility parameter $WL/2K_\phi$. As indicated, the increase varies approximately linearly with the base flexibility parameter for the practical range shown.

A present-day large booster has a value of W_{CR}/W approximately equal to 7 and a base flexibility parameter $WL/2K_\phi$ equal to 0.04. The increase in bending moment due to beam columning with a rigid base is 13.0%. Considering base flexibility, the increase becomes 18.4%. Increases of these magnitudes certainly warrant as much consideration in analysis as do the wind-induced oscillation effects.

The significance of the base flexibility parameter $WL/2K_\phi$ can be determined by consideration of the free vibrations of the beam shown in Fig. 1, with an infinitely rigid structure and $\phi_0 = 0$. Upon calculating the natural frequency of this system (assuming small displacements), one finds

$$\omega = \left[\frac{3g}{2L}\left(\frac{1}{WL/2K_\phi} - 1\right)\right]^{1/2} \quad (10)$$

where ω is the natural frequency and g is the acceleration due to gravity. Note that, as $WL/2K_\phi$ approaches unity, the vertical position of the beam becomes unstable, and large amplitudes are induced. This is equivalent to stating [in Eq. (9)] that the base moments will become very large as $WL/2K_\phi$ approaches unity.

Before concluding, the relationship between beam columning on the launch pad and in flight will be considered briefly. Unpublished derivations by the author have shown that increases in flight bending moments due to beam columning, for a uniform vehicle, can be approximated by

$$\frac{M(L/2)}{M_{\text{rigid}}} \cong \frac{1}{1 - (T/T_{CR})} \quad (11)$$

where $M(L/2)$ is the bending moment at the vehicle's center, M_{rigid} is the rigid bending moment, T is the thrust, and T_{CR} is the critical thrust force, given by

$$T_{CR} = \tfrac{96}{5}(EI/L^2) \quad (12)$$

It was shown earlier that the beam columning increases on the launch stand are a function of W/W_{CR}. To approximate the relative severity of free-free beam columning as compared to that on the launch stand, assume that it can be defined adequately by the relationship $(T/T_{CR})/(W/W_{CR})$. With the aid of Eqs. (5) and (11), one has

$$\frac{T/T_{CR}}{W/W_{CR}} \cong 0.408\frac{T}{W} \quad (13)$$

This equation essentially demonstrates the effects of the thrust-to-weight ratio on the relative severity of in-flight beam columning action. For most contemporary liquid boosters with a T/W ratio of 1.25, Eq. (13) would indicate that the free-free beam columning increase is only one-half that of the fixed-free condition. This has been found to be the case in a large booster, which experiences a 13% increase on the launch pad (with $K_\phi = \infty$) and a 6 to 7% increase in flight. As the thrust-to-weight ratio increases, the relative severity of the flight beam columning becomes more prominent.

References

[1] Ezra, A. A. and Birnbaum, S., "Design criteria for space vehicles to resist wind induced oscillations," ARS J. **31,** 1764–1766 (1961).

[2] Bohne, Q. R., "Power spectral density techniques in ground wind drag analyses," ARS J. **32,** 227–233 (1962).

[3] Watson, G. N., *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, Great Britain, 1958), Table III.

[4] Timoshenko, S., *Theory of Elastic Stability* (McGraw-Hill Book Co., New York, 1936), p. 118.

Wave Reflection from the Intersection of Oblique Shock Waves of the Same Family

Glen W. Zumwalt* and John J. Flynn Jr.†
Oklahoma State University, Stillwater, Okla.

Nomenclature

M = Mach number
p = static pressure
γ = specific heat ratio
δ = flow deflection angle for oblique shock
θ = flow angle
σ = oblique shock angle

Subscripts

1, 2, 3, 4 = regions defined by Fig. 1

When oblique plane shock waves intersect and coalesce, as shown in Fig. 1, a "three-shock system" results. At the intersection of the shocks, a slip line is produced, and a weak wave may be propagated into the two-shock side (region 3). This reflected wave can be either expansive or compressive, depending upon the combination of M_1, δ_1, and δ_2.

To determine which type of wave will be reflected, an investigation was carried out for two equal turn angles, $\delta_1 = \delta_2 = \delta$, and various Mach numbers, M_1, for perfect gas γ values of 1.28, 1.40, and 1.67. This was performed by digital computer solution of the "nonreflection" case, i.e., the combination of M_1 and δ which gives a zero-strength reflected wave. The requirement for this is that $p_3 = p_4$ and $\theta_3 = \theta_4$.

The oblique shock relations are

$$\frac{p_2}{p_1} = \frac{2\gamma M_1^2 \sin^2\sigma_1 - (\gamma - 1)}{\gamma + 1} \tag{1}$$

$$M_2^2 = \frac{(\gamma + 1)^2 M_1^4 \sin^2\sigma_1 - 4(M_1^2 \sin^2\sigma_1 - 1)(\gamma M_1^2 \sin^2\sigma_1 + 1)}{[2\gamma M_1^2 \sin^2\sigma_1 - (\gamma - 1)][(\gamma - 1)M_1^2 \sin^2\sigma_1 + 2]} \tag{2}$$

Received March 27, 1963; revision received June 21, 1963. This note incorporated material from the M. S. Report of the second author at Oklahoma State University, Stillwater, Okla.

* Associate Professor of Mechanical and Aerospace Engineering. Member AIAA.

† Graduate Student, School of Mechanical Engineering; also Captain, U. S. Air Force.

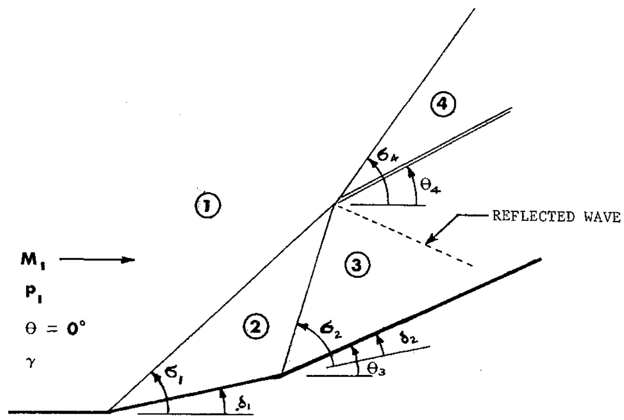

Fig. 1 Model for the three-shock system.

For the relation between turning angle δ and shock angle α, Thompson's cubic equation in $\sin^2\alpha$ is used[1]:

$$\sin^6\sigma_1 + b \sin^4\alpha_1 + c \sin^2\sigma_1 + d = 0 \tag{3}$$

where

$$b = -[(M_1^2 + 2)/M_1^2] - \gamma \sin^2\delta_1$$

$$c = \frac{2M_1^2 + 1}{M_1^4} + \left[\frac{(\gamma + 1)^2}{4} + \frac{(\gamma - 1)}{M_1^2}\right] \sin^2\delta_1$$

$$d = -\cos^2\delta_1/M_1^4$$

Similar relations, of course, exist for the shocks between regions 2 and 3 and regions 1 and 4. For the cubic equation, three real roots exist for $\sin^2\sigma$. The smallest results in an expansion shock, violating the second law of thermodynamics, and must be disregarded. The largest root corresponds to the strong shock solution, which will not exist as a plane shock for such a geometry. The middle value is the root corresponding to the weak shock and is the one having physical significance.

A Fortran program was prepared for the IBM 1620 computer to find by interaction the "no-reflection" condition. The sign of the derivative $(d/dM_1)/(p_3/p_4)$ was determined by the computer. If the derivative has a positive sign, higher

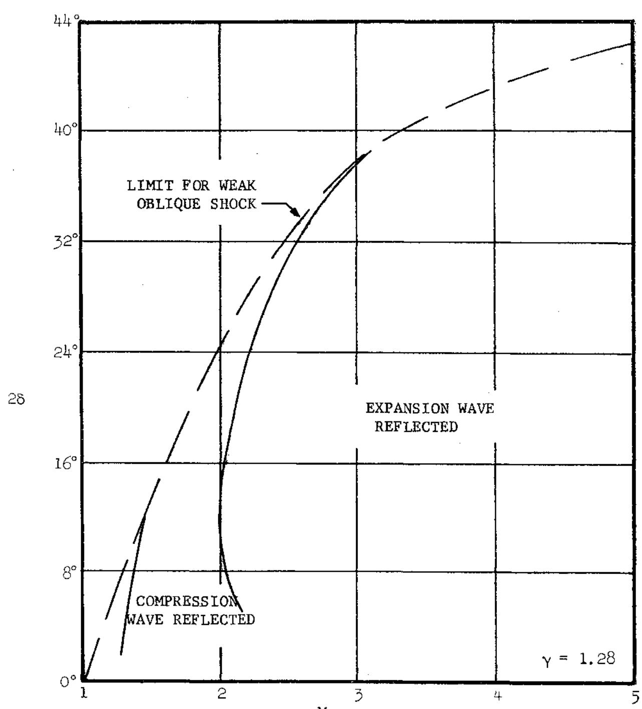

Fig. 2 Conditions for no-reflected wave from the shock intersection (γ = 1.28).